ON THE INVARIANT REPRESENTATION OF SPIN TENSORS WITH APPLICATIONS

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Abstract—The invariant representation of the spin tensor defined as the rotation rate of a principal triad for a symmetric and non-degenerate tensor is derived on the basis of the general solution of a linear tensorial equation. The result can be naturally specified to study the spin of the stretch tensors and to investigate the relations between various rotation rate tensors encountered frequently in modern continuum mechanics. A remarkable formula which relates the generalized stress conjugate to the generalized strain in Hill's sense, to Cauchy stress, is obtained in invariant form through the work conjugate principle. Particularly, a detailed discussion on the time rate of logarithmic strain and its conjugate stress is made as the principal axes of strain are not fixed during deformation.

1. INTRODUCTION

The method of principal axes developed by Hill (1970, 1978) is well-known and proved to be very influential and prominent in modern continuum mechanics. The essence of Hill's method is to seek a representation of tensors with respect to the trace axes of deformation. Hill holds that this method is a sure way to avoid the labyrinthine complexity encountered in tensor algebra. He also provided a representation of a spin tensor, which is given in a component form with respect to a fixed background triad. Furthermore, this principal axis method has found its many important applications in studying and formulating constitutive relations, one of which could be connected with the generalized strain measure and the corresponding work conjugate stress. Specifically, an interesting example in these applications could be referred to the logarithmic strain and its conjugate stress. Since Hencky first introduced the logarithmic strains referred to as "natural" or "true" ones, they have been favoured in metallurgical and material science literature. However, since then this strain measure has not yet found its uses in the case when the principal axes of strain are not fixed. Truesdell and Toupin (1960) have taken note of this situation and Hill (1970) also argued the inherent advantages of using the logarithmic strain measure in certain constitutive inequalities. This problem has been discussed by Hutchinson and Neale (1980), Stören and Rice (1975) and others. They all confirmed that the logarithmic strain is useful in formulating the finite theory of plasticity. However, as pointed out by Stören and Rice, the strain ln U is essentially intractable as strain measure, where U is the right stretch tensor. After discussing the tensorial Hencky measure of strain and strain rate for finite deformation, Fitzgerald (1980) subjectively concluded that the use of the logarithmic strain is only limited to the problems with fixed principal strain axes. On the other hand, using the logarithmic strain measure, Gurtin and Spear (1983) obtained a relationship between the logarithmic strain rate and the stretching. Recently, Hoger (1986, 1987) derived an expression for the time rate of ln U, based on which a properly invariant representation for the corresponding stress conjugate to the logarithmic strain has been derived. From what is mentioned here, there is still a need to make things more clear in dealing with the logarithmic strain, its time rate and the associated conjugate stress when the principal axes of strain are not fixed.

It should be pointed out that the representation of a spin given in component form by Hill in his early study might not be convenient for the purpose of theoretical study and it does not reflect the harmony with tensor analysis as well. Considering this fact, it is necessary to seek the invariant representation of spin tensors and this is our main aim in the present paper. As shown in the following section, the approach developed will differ from that offered by Mehrabadi and Nemat-Nasser (1987). In our approach, we first establish a general tensorial equation to determine the spin of a symmetric tensor, and then solve the equation by an expansion technique in terms of a group of complete and irreducible generators. The rate of a symmetric tensor is supposed to be divided into two parts, one represents the contribution due to the rotation of the principal axes, the other called the rate of a tensor with its principal axes fixed is objective. In this way, the influence of the rotation of the principal axes on the rate of the tensor can be eliminated. In the third section, we shall focus on the detailed discussion of the stretch tensor and its spin, using the general results obtained in the second section. In the fourth section, the general relationship between the rates of generalized strains and stretch tensor, its time rate as well as the spin, is established. Particularly, the relation between the logarithmic strain rate and the right stretch tensor and its spin is given in a compact form. In the fifth section, the problem regarding work conjugate and conjugate stress is discussed in detail and the general relations between Cauchy stress and the stress conjugate to the generalized strain are derived in closed forms.

2. INVARIANT REPRESENTATION OF A SPIN

Suppose there is a configuration B in three-dimensional space of which a point is denoted by X. Let U be a symmetric and non-degenerate tensor attached to this point X at time t, that is, $\det U \neq 0$, at any time t. The spectral decomposition of U can be written in the form

$$U = \sum_{i=1}^{3} \lambda_i N_i \otimes N_i,$$ (1)

where $\{\lambda_i\}_{i=1,2,3}$ are assumed to be the three distinct eigenvalues of U, $\{N_i\}_{i=1,2,3}$ are the corresponding eigenvectors, which form a local orthonormal triad called the principal triad of U. This triad rotates against a fixed background one and its spin Ω^L is defined by

$$\Omega^L = \sum_{i=1}^{3} \dot{N}_i \otimes N_i \quad \text{or equivalently} \quad \dot{N}_i = \Omega^L \cdot N_i,$$ (2)

where $\dot{N}_i$ denotes the time derivative of N_i at X fixed. Because of the orthonormality of the principal triad, Ω^L is a skew tensor

$$\Omega^L = -(\Omega^L)^\tau,$$ (3)

where $(\ )^\tau$ means the transpose of the tensor in the parentheses.

Hill (1978) found a representation for Ω^L which is given in component form

$$\Omega^L = \omega_{ij}^L N_i \otimes N_j$$ (4a)

and

$$\omega_{ij}^L = \frac{\dot{\lambda}_{ij}}{\lambda_j - \lambda_i}, \quad \text{(no sum over } i, j \text{ and } \lambda_i \neq \lambda_j)$$ (4b)

where λ_{ri} represents the components of the tensor $\dot{U}$ based on the Lagrangian triad,

$$\dot{U} = \lambda_{rs} N_r \otimes N_s.$$

Obviously eqn (4a) is not an absolute representation of Ω^L. The purpose of the present paper is mainly to seek various invariant representations of Ω^L and their applications.

First, let us consider the skew tensor $(\dot{U}U^{-1} - U^{-1}\dot{U})$. From eqns (1) and (2), we derive

$$2\Omega^L - U\Omega^L U^{-1} - U^{-1}\Omega^L U = \dot{U}U^{-1} - U^{-1}\dot{U}. \tag{5}$$

This equation can be considered as a linear tensorial equation for Ω^L if U and $\dot{U}$ are known. Since Ω^L is the spin of the principal triad of U, all isotropic tensor functions f(U) should share the common principal triad. Therefore, replacing U by f(U) in (5) we have

$$2\Omega^L - f(U)\Omega^L f^{-1}(U) - f^{-1}(U)\Omega^L f(U) = \dot{f}(U)f^{-1}(U) - f^{-1}(U)\dot{f}(U). \tag{6}$$

Obviously, we can see that (6) has the same form as (5), therefore, the solution of (6) must be the same as that of (5). For instance, if we select either $f(U) = C = U^2$ or the deviatoric tensor U' of U, that is, $f(U) = U' = U - 1/3(\text{tr } U)I$, eqn (6) directly becomes

$$2\Omega^L - C\Omega^L C^{-1} - C^{-1}\Omega^L C = \dot{C}C^{-1} - C^{-1}\dot{C}, \tag{7a}$$

$$2\Omega^L - U'\Omega^L U'^{-1} - U'^{-1}\Omega^L U' = \dot{U}'U'^{-1} - U'^{-1}\dot{U}'. \tag{7b}$$

An elegant technique to solve a family of linear tensorial equation which are more general than (5) has been recently presented by Wang and Duan (1989). The key of this technique is to expand the solution in terms of a group of complete and irreducible generators associated with this tensorial equation. Based on this method and considering the fact that Ω^L is an isotropic skew tensor function of U and $\dot{U}$, we can easily find the following three linearly independent generators of Ω^L

$$\Omega^{(1)} = U\dot{U} - \dot{U}U, \quad \Omega^{(0)} = U\dot{U}U^{-1} - U^{-1}\dot{U}U, \quad \Omega^{(-1)} = \dot{U}U^{-1} - U^{-1}\dot{U}, \tag{8}$$

when U has three distinct eigenvalues. Further, the solution of (5) can be expressed in terms of these generators,

$$\Omega^L = \sum_{i=-1}^{1} \omega_i \Omega^{(i)} \quad \text{with } \omega_i = \omega_i(I, II, III), \tag{9}$$

where I, II and III are, respectively, the invariants of U, and the scalars ω_i ($i = -1, 0, 1$) are their functions, which need to be determined. In fact, by substituting (9) into (5) and making use of the Cayley–Hamilton theorem, we can transform (5) into the following form

$$[3III\omega_1 + (I^2 - II)\omega_0 + I\omega_{-1}]\Omega^{(1)} + [-I III\omega_1 + (3III - I II)\omega_0 - II\omega_{-1}]\Omega^{(0)}$$
$$+ [II III\omega_1 + (II^2 - I III)\omega_0 + 3III\omega_{-1}]\Omega^{(-1)} = III\Omega^{(-1)}. \tag{10}$$

Since the three generators $\Omega^{(i)}$ are linearly independent, we can easily find from (10) the solutions for ω_i

$$\omega_1 = \frac{1}{\Delta}(II - 3I III), \quad \omega_0 = \frac{III}{\Delta}(3II - I^2), \quad \omega_{-1} = \frac{III}{\Delta}(I^3 - 4I II + 9III) \tag{11}$$

with

$$\Delta = 4I^3 III - I^2 II^2 - 18I\,II\,III + 4II^3 + 27III^2. \tag{12}$$

It can be also expressed in a compact form

$$\Delta = 4II_{U'}^3 + 27III_{U'}^2, \tag{13}$$

where $II_{U'}$ and $III_{U'}$ are, respectively, the invariants of the deviatoric tensor U'. Finally, from (5), (9) and (11) we find

$$\Omega^L = \frac{1}{\Delta}\{(II^2 - 3I\,III)(U\dot{U} - \dot{U}U) + III(3II - I^2)(U\dot{U}U^{-1} - U^{-1}\dot{U}U)$$

$$+ III(I^3 - 4I\,II + 9III)(\dot{U}U^{-1} - U^{-1}\dot{U})\}. \tag{14}$$

This formula is valid when $\Delta \neq 0$. The condition $\Delta = 0$ holds if, and only if, U owns multiple eigenvalues. The proof is as follows.

Obviously we have

$$I_{U'} = \lambda_1' + \lambda_2' + \lambda_3' = 0, \quad II_{U'} = -(\lambda_1'\lambda_2' + \lambda_1'^2 + \lambda_2'^2), \quad III_{U'} = -\lambda_1'\lambda_2'(\lambda_1' + \lambda_2'). \tag{15}$$

Through some algebraic operation we obtain

$$\Delta = -[(\lambda_1' - \lambda_2')(\lambda_2' - \lambda_3')(\lambda_3' - \lambda_1')]^2. \tag{16}$$

When U has two multiple eigenvalues, $\Omega^{(i)}$ ($i = -1, 0, 1$) are not of linear independence. In the case $\lambda = \lambda_1 = \lambda_2 \neq \lambda_3$, using the minimal polynomial of U

$$U^2 - (\lambda + \lambda_3)U + \lambda\lambda_3 I = 0, \tag{17}$$

one can show that there is only one independent generator of the spin Ω^L, say $\Omega^{(-1)}$, then from (5) and (17) we obtain

$$\Omega^L = -\frac{\lambda\lambda_3}{(\lambda - \lambda_3)^2}(\dot{U}U^{-1} - U^{-1}\dot{U}), \tag{18}$$

equivalently,

$$\Omega^L = \dot{N}_3 \otimes N_3 - N_3 \otimes \dot{N}_3. \tag{19}$$

Ω^L can be also obtained from (14) through a limit process as Δ tends to zero. The limit result depends on the process and might differ from (18) by a term $a(N_1 \otimes N_2 - N_2 \otimes N_1)$. However, this term is not essential to Ω^L.

If U has three multiple eigenvalues, it is a spherical tensor and any orthonormal triad can be taken as the principal one. In this case, Ω^L could be any skew tensor. This result can be derived from (14) by assuming a limit process $\lambda_1 \to \lambda_2 \to \lambda_3 = \lambda$. In brief, (14) can be applied for arbitrary distribution of eigenvalues of U.

We should mention that the selection of the three independent generators as given in (8) is by no means unique. For example, instead of (8), one can choose

$$\Omega^{(1)} = U\dot{U} - \dot{U}U, \quad \Omega^{(2)} = U^2\dot{U} - \dot{U}U^2, \quad \Omega^{(3)} = U^2\dot{U}U - U\dot{U}U^2 \tag{20}$$

as the independent generators of Ω^L. By making use of Cayley–Hamilton theorem,

$$U^{-1} = III^{-1}(U^2 - IU + II\,I), \tag{21}$$

eqn (14) can be easily changed into the form

$$\Omega^L = \frac{1}{\Delta}[(I^4 - 5 I^2 II + 6 I III + 4 II^2)(U\dot{U} - \dot{U}U) + (4 I II - I^3 - 9 III)(U^2\dot{U} - \dot{U}U^2)$$

$$+ (I^2 - 3 II)(U^2\dot{U}U - U\dot{U}U^2)]. \quad (22)$$

Substituting U and I. II as well as III by U' and its corresponding invariants $I_{U'} = 0$, $II_{U'}$ and $III_{U'}$, the above formula (22) is completely equivalent to that obtained by Mehrabadi and Nemat-Nasser (1987).

3. STRETCH TENSORS AND THEIR SPINS

Let X denote the position of a moving material particle P at a reference (Lagrangian) configuration, the motion of the material particle can be described by $x = x(X, t)$ where x represents the position of P at time t in Eulerian configuration. The deformation gradient F is defined by

$$dx = F(X, t) dX. \quad (23)$$

F is not degenerated and $III = \det F > 0$. According to the polar decomposition theorem, we have

$$F = RU = VR, \quad (24)$$

where R, U and V are the rotation tensor, right and left stretch tensors, respectively. For later convenience, we introduce a local rotation transformation as

$$R: \quad S \to \tilde{S} = RSR^T \quad (25a)$$

which maps any second rank tensor S given in the reference configuration into the current configuration. Therefore, the inverse transformation R^{-1} maps a tensor D given in the current configuration into the reference configuration

$$R^{-1}: \quad D \to \hat{D} = R^T DR. \quad (25b)$$

So in the following study we shall focus our attention on the problem in the reference configuration, the results can be readily transferred to the current configuration through the transformation R.

According to Hill, a family of strain tensors is defined by

$$E = f(U) := \sum_{i=1}^{3} f(\lambda_i)N_i \otimes N_i, \quad (26)$$

where E called the generalized strain tensor is an isotropic tensorial function of U, and the corresponding scalar function $f = f(\lambda)$ is smooth, monotonic and satisfies the following conditions

$$f(1) = 0, \quad f'(1) = 1. \quad (27)$$

This family of strain measure includes the Seth strain

$$E^{(n)} = \frac{1}{2n}[U^{2n} - 1]. \quad (28)$$

As n takes the value $n = 1, -1, \frac{1}{2}$, (28) gives, respectively, the Green strain, the Almansi strain and the stretch strain measures. Particularly, when $n \to 0$, (28) corresponds to the logarithmic strain

$$E^{(0)} = \ln U, \tag{29}$$

this is sometime called the generalized Hencky strain.

Let $v = \dot{x}(X, t)$ denote the velocity of the material point P at time t, the velocity gradient tensor L is defined by

$$dv = L\,dx \quad \text{and} \quad L = \dot{F}F^{-1}. \tag{30a,b}$$

where ($\dot{}$) means the time derivative with respect to t at X fixed. Usually L can be additively decomposed into two parts

$$L = D + W \tag{31}$$

with

$$D = \tfrac{1}{2}(L + L^T), \quad W = \tfrac{1}{2}(L - L^T). \tag{32}$$

where the symmetric part D and the skew part W are, respectively, called the stretching and the material rotation rate tensor. Combining (24) and (30b) with (31)–(32) we obtain

$$L = \Omega^R + R\dot{U}U^{-1}R^T,$$
$$D = \tfrac{1}{2}R(\dot{U}U^{-1} + U^{-1}\dot{U})R^T = R\hat{D}R^T,$$
$$W = \Omega^R + \tfrac{1}{2}R(\dot{U}U^{-1} - U^{-1}\dot{U})R^T, \tag{33}$$

with

$$\Omega^R = \dot{R}R^T \quad \text{and} \quad \hat{D} = \tfrac{1}{2}(\dot{U}U^{-1} + U^{-1}\dot{U}) \tag{34a,b}$$

representing the relative rotation rate and the stretching in the reference configuration. If N_i and n_i ($i = 1, 2, 3$) are the eigenvectors of U and V, respectively, then the spins Ω^L of U and Ω^E of V are defined by

$$\dot{N}_i = \Omega^L \cdot N_i, \quad \dot{n}_i = \Omega^E \cdot n_i, \tag{35}$$

or equivalently

$$\Omega^L = \sum_{i=1}^{3} \dot{N}_i \otimes N_i, \quad \Omega^E = \sum_{i=1}^{3} \dot{n}_i \otimes n_i. \tag{36}$$

Based on the previous discussion, the relationship between Ω^L, U and $\dot{U}$ is completely the same as eqn (5), of which the solution is given by (14). Since $V = FR^T = RUR^T$, we can prove $n_i = RN_i$, thus

$$\Omega^E = \Omega^R + \tilde{\Omega}^L \quad \text{and} \quad \tilde{\Omega}^L = R\Omega^L R^T. \tag{37a,b}$$

From (5) and (33), it is found that

$$W = \Omega^R + R[\Omega^L - \tfrac{1}{2}(U\Omega^L U^{-1} + U^{-1}\Omega^L U)]R^T, \tag{38a}$$

then, using (37) and the above formula, we obtain

$$W = \Omega^E - \tfrac{1}{2}[V\Omega^L V^{-1} + V^{-1}\Omega^L V]. \tag{38b}$$

In what follows, we would like to express $\dot{U}$ as a function of $\hat{D}$ and U. This can be

done by solving the linear tensorial equation (37b). Since $\dot{\mathbf{U}}$ is symmetric and linear with respect to $\hat{\mathbf{D}}$, there are only six generators for $\dot{\mathbf{U}}$, that is $\hat{\mathbf{D}}$, $\mathbf{U}\hat{\mathbf{D}}\mathbf{U}$, $\mathbf{U}^2\hat{\mathbf{D}}\mathbf{U}^2$, $\mathbf{U}\hat{\mathbf{D}}+\hat{\mathbf{D}}\mathbf{U}$, $\mathbf{U}^2\hat{\mathbf{D}}+\hat{\mathbf{D}}\mathbf{U}^2$ as well as $\mathbf{U}^2\hat{\mathbf{D}}\mathbf{U}+\mathbf{U}\hat{\mathbf{D}}\mathbf{U}^2$. As U has three distinct eigenvalues, these generators are obviously complete and irreducible. Therefore, by a similar procedure to that which we followed to deal with eqn (5) and taking $\dot{\mathbf{U}}$ as the linear combination of the above listed generators, we obtain from eqn (34b)

$$\dot{\mathbf{U}} = \frac{1}{\mathrm{I\,III-III}}[\mathrm{I\,III}\hat{\mathbf{D}}+(\mathrm{I}^2+\mathrm{II})\mathbf{U}\hat{\mathbf{D}}\mathbf{U}+\mathbf{U}^2\hat{\mathbf{D}}\mathbf{U}^2-\mathrm{III}(\mathbf{U}\hat{\mathbf{D}}+\hat{\mathbf{D}}\mathbf{U})-\mathrm{I}(\mathbf{U}^2\hat{\mathbf{D}}\mathbf{U}+\mathbf{U}\hat{\mathbf{D}}\mathbf{U}^2)].$$

(39)

This expression presents the same result as given by Mehrabadi and Nemat-Nasser (1987) with the exception of a misprint there.

Since $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^\mathrm{T}$, we can easily prove

$$\mathbf{V}^O := \mathbf{V}-\mathbf{\Omega}^R\mathbf{V}+\mathbf{V}\mathbf{\Omega}^R = \mathbf{R}\dot{\mathbf{U}}\mathbf{R}^\mathrm{T},$$

(40)

where $\mathbf{V}^O$ can be called the relative time derivative of V, and it is objective. Substituting (42) into (43), it can be shown that

$$\mathbf{V}^O = \frac{1}{\mathrm{I\,III-III}}[\mathrm{I\,III}\mathbf{D}+(\mathrm{I}^2+\mathrm{II})\mathbf{V}\mathbf{D}\mathbf{V}+\mathbf{V}^2\mathbf{D}\mathbf{V}^2-\mathrm{III}(\mathbf{V}\mathbf{D}+\mathbf{V}\mathbf{U})-\mathrm{I}(\mathbf{V}^2\mathbf{D}\mathbf{V}+\mathbf{V}\mathbf{D}\mathbf{V}^2)].$$

(41)

Similarly, a representation for $\mathbf{\Omega}^L$ in terms of U and $\hat{\mathbf{D}}$ can be directly found through (22) and (39), and it takes the form

$$\mathbf{\Omega}^L = p_1(\mathbf{U}\hat{\mathbf{D}}-\hat{\mathbf{D}}\mathbf{U})+p_0(\mathbf{U}\hat{\mathbf{D}}\mathbf{U}^{-1}-\mathbf{U}^{-1}\hat{\mathbf{D}}\mathbf{U})+p_{-1}(\hat{\mathbf{D}}\mathbf{U}^{-1}-\mathbf{U}^{-1}\hat{\mathbf{D}}),$$

(42a)

where

$$p_1 = \frac{2}{\Delta(\mathrm{I\,II-III})}(\mathrm{I}^2\mathrm{III}^2-4\mathrm{I\,II}^2\mathrm{III}+6\mathrm{II\,III}^2+\mathrm{II}^4),$$

$$p_0 = \frac{-2\mathrm{III}}{\Delta(\mathrm{I\,II-III})}(\mathrm{I}^3\mathrm{III}-7\mathrm{I\,II\,III}+9\mathrm{III}^2+\mathrm{II}^3),$$

$$p_{-1} = \frac{2\mathrm{III}^2}{\Delta(\mathrm{I\,II-III})}(\mathrm{I}^4-4\mathrm{I}^2\mathrm{II}+6\mathrm{I\,III}+\mathrm{II}^2).$$

(42b)

From (37) and (42a,b), $\mathbf{\Omega}^E$ can be expressed by

$$\mathbf{\Omega}^E = \mathbf{\Omega}^R+p_1(\mathbf{V}\mathbf{D}-\mathbf{D}\mathbf{V})+p_0(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}-\mathbf{V}^{-1}\mathbf{D}\mathbf{V})+p_{-1}(\mathbf{D}\mathbf{V}^{-1}-\mathbf{V}^{-1}\mathbf{D}).$$

(43)

These two representations (42a,b) and (43) have not been obtained before.

As described by Gurtin and Spear (1983), the co-rotational derivative of U, that is, the time derivative of U with respect to its principal triad fixed can be defined as

$$\mathbf{U}^* = \dot{\mathbf{U}}-\mathbf{\Omega}^L\mathbf{U}+\mathbf{U}\mathbf{\Omega}^L = \sum_{i=1}^{3} \dot{\lambda}_i\mathbf{N}_i \otimes \mathbf{N}_i.$$

(44)

Similarly, the co-rotational derivatives for an isotropic tensorial function $f(\mathbf{U})$ is given by

$$[f(\mathbf{U})]^* = \dot{f}(\mathbf{U})-\mathbf{\Omega}^L f(\mathbf{U})+f(\mathbf{U})\mathbf{\Omega}^L.$$

(45)

These co-rotational derivatives defined in (44) and (45) are objective in the sense that

the influences due to the rotation of the principal axis on U or on $f(U)$ have been eliminated. If we rewrite (44) in the form

$$\dot{U} = \tfrac{1}{3}(\text{tr }\dot{U})I + \sum_{i=1}^{3} \dot{\lambda}'_i N_i \otimes N_i + \Omega^L U - U\Omega^L, \tag{46}$$

where $\dot{\lambda}'_i = \dot{\lambda}_i - 1/3(\dot{\lambda}_1 + \dot{\lambda}_2 + \dot{\lambda}_3)$. It is seen that $\dot{U}$ can be additively decomposed into three parts, the first two parts representing, respectively, the change rate of tr U and the deviatoric change rate of U watched by an observor fixed on the principal triad. This decomposition might be useful in constructing the constitutive relations for rate-dependent and hypoelastic materials.

Finally, it should be mentioned that the above discussion can remain the same for the symmetric left stretch tensor V given in current configuration. In fact, the spin of V is Ω^E, therefore, its co-rotational derivative V^* is defined as

$$V^* = \dot{V} - \Omega^E V + V\Omega^E = V^O - \tilde{\Omega}^L V + V\tilde{\Omega}^L, \tag{47}$$

we see from (47) that the co-rotational derivative of V is in general not the same as the relative time derivative of it.

4. RATES OF GENERALIZED STRAINS

The material derivative of a generalized strain tensor E defined in (29) can be directly calculated by

$$\dot{E} = \sum_{i=1}^{3} \dot{\lambda}_i g(\lambda_i) N_i \otimes N_i + \Omega^L E - E\Omega^L, \tag{48}$$

where $g(\lambda) = df/d\lambda$ and $g(1) = 1$. Using (44) and the definition of $g(U)$, the above formula can be transferred into its absolute representation

$$\dot{E} = \tfrac{1}{2}[g(U)\dot{U} + \dot{U}g(U)] - \tfrac{1}{2}[g(U)(\Omega^L U - U\Omega^L) + (\Omega^L U - U\Omega^L)g(U)] + \Omega^L E - E\Omega^L. \tag{49}$$

Therefore, it leads to

$$\dot{E}^{(n)} = \tfrac{1}{2}(U^{2n-1}\dot{U} + \dot{U}U^{2n-1}) + \frac{1-n}{2n}(\Omega^L U^{2n} - U^{2n}\Omega^L) - \tfrac{1}{2}(U^{2n-1}\Omega^L U - U\Omega^L U^{2n-1}) \tag{50}$$

for the rate of Seth strain. For $n = 0$, by carrying out a limit process on (50), or from (49), it is easy to obtain

$$\dot{E}^{(0)} = \hat{D} + \tfrac{1}{2}(U\Omega^L U^{-1} - U^{-1}\Omega^L U) + \Omega^L \ln U - \ln U\Omega^L. \tag{51}$$

It would be sometimes convenient to express the quantity $1/2(U\Omega^L U^{-1} - U^{-1}\Omega^L U)$ in terms of $\hat{D}$ and U through the fundamental solution (22). To do this we find

$$\tfrac{1}{2}(U\Omega^L U^{-1} - U^{-1}\Omega^L U) = \frac{1}{\Delta} [d_0\hat{D} + q_{-2}U^{-1}\hat{D}U^{-1} + \tfrac{1}{2}q_{-1}(\hat{D}U^{-1} + U^{-1}\hat{D})$$
$$+ \tfrac{1}{2}q_0(U\hat{D}U^{-1} + U^{-1}\hat{D}U) + \tfrac{1}{2}q_1(U\hat{D} + \hat{D}U) + q_2 U\hat{D}U], \tag{52}$$

where

$$d_0 = -(\mathrm{I}^3\,\mathrm{III} - 7\,\mathrm{I}\,\mathrm{II}\,\mathrm{III} + 9\,\mathrm{III}^2 + \mathrm{II}^3), \quad q_{-2} = \mathrm{III}^2(3\,\mathrm{II} - \mathrm{I}^2),$$

$$q_{-1} = (\mathrm{I}^2\,\mathrm{II} - 3\,\mathrm{I}\,\mathrm{III} - 2\,\mathrm{II}^2)\mathrm{III}, \quad\quad q_0 = \mathrm{III}(9\,\mathrm{III} - \mathrm{I}\,\mathrm{II}),$$

$$q_1 = \mathrm{I}\,\mathrm{II}^2 - 2\,\mathrm{I}^2\,\mathrm{III} - 3\,\mathrm{II}\,\mathrm{III}, \quad\quad q_2 = 3\,\mathrm{I}\,\mathrm{III} - \mathrm{II}^2.$$

After a simple algebraic calculation we can also arrive at

$$(\ln \mathbf{U})^* = \sum_{i=1}^{3} \frac{\dot\lambda_i}{\lambda_i}\mathbf{N}_i \otimes \mathbf{N}_i = \hat{\mathbf{D}} + \tfrac{1}{2}(\mathbf{U}\boldsymbol{\Omega}^L\mathbf{U}^{-1} - \mathbf{U}^{-1}\boldsymbol{\Omega}^L\mathbf{U}). \tag{53}$$

Combining (52) with (53), the co-rotational derivative $(\ln \mathbf{U})^*$ is expressed by $\hat{\mathbf{D}}$ and $\mathbf{U}$, which is useful in determining the conjugate stress of the generalized Hencky strain.

We insert eqn (53) into (48) and arrive at

$$\dot{\mathbf{E}} = g(\mathbf{U})\mathbf{U}(\ln \mathbf{U})^* + \boldsymbol{\Omega}^L\mathbf{E} - \mathbf{E}\boldsymbol{\Omega}^L = \tfrac{1}{2}[g(\mathbf{U})\mathbf{U}(\ln \mathbf{U})^* + (\ln \mathbf{U})^*\mathbf{U}g(\mathbf{U})] + \boldsymbol{\Omega}^L\mathbf{E} - \mathbf{E}\boldsymbol{\Omega}^L, \tag{54}$$

which leads to the following expression

$$\dot{\mathbf{E}}^{(n)} = \tfrac{1}{2}[\mathbf{U}^{2n}(\ln \mathbf{U})^* + (\ln \mathbf{U})^*\mathbf{U}^{2n}] + \frac{1}{2n}(\boldsymbol{\Omega}^L\mathbf{U}^{2n} - \mathbf{U}^{2n}\boldsymbol{\Omega}^L), \tag{55}$$

for the Seth strain $\mathbf{E}^{(n)}$. In particular, we obtain

$$\dot{\mathbf{E}}^{(0)} = (\ln \mathbf{U})^* + \boldsymbol{\Omega}^L\mathbf{E}^{(0)} - \mathbf{E}^{(0)}\boldsymbol{\Omega}^L. \tag{56}$$

In what follows, we want to seek the expressions for the relative time derivative and the co-rotational derivative of $\ln \mathbf{V}$. To do this, we first make use of $f(\mathbf{V}) = \mathbf{R}f(\mathbf{U})\mathbf{R}^{\mathrm{T}}$, and have

$$\frac{\mathrm{d}f(\mathbf{V})}{\mathrm{d}t} = \mathbf{R}\frac{\mathrm{d}f(\mathbf{U})}{\mathrm{d}t}\mathbf{R}^{\mathrm{T}} + \boldsymbol{\Omega}^R f(\mathbf{V}) - f(\mathbf{V})\boldsymbol{\Omega}^R. \tag{57}$$

Combining (57) with equations (51) and (47), finally we arrive at

$$(\ln \mathbf{V})' = \mathbf{D} + \tfrac{1}{2}(\mathbf{V}\tilde{\boldsymbol{\Omega}}^L\mathbf{V}^{-1} - \mathbf{V}^{-1}\tilde{\boldsymbol{\Omega}}^L\mathbf{V}) + \tilde{\boldsymbol{\Omega}}^L \ln \mathbf{V} - \ln \mathbf{V}\tilde{\boldsymbol{\Omega}}^L, \tag{58}$$

$$(\ln \mathbf{V})^* = \mathbf{D} + \tfrac{1}{2}(\mathbf{V}\tilde{\boldsymbol{\Omega}}^L\mathbf{V}^{-1} - \mathbf{V}^{-1}\tilde{\boldsymbol{\Omega}}^L\mathbf{V}) = \mathbf{D} + \tfrac{1}{2}(\mathbf{F}\boldsymbol{\Omega}^L\mathbf{F}^{-1} - \mathbf{F}^{-\mathrm{T}}\boldsymbol{\Omega}^L\mathbf{F}^{\mathrm{T}}), \tag{59}$$

where $\tilde{\boldsymbol{\Omega}}^L = \mathbf{R}\boldsymbol{\Omega}^L\mathbf{R}^{\mathrm{T}}$ as defined in (37b). The result shown in (59) is the same in form as obtained by Gurtin and Spear (1983), where they did not give an explicit representation for $\boldsymbol{\Omega}_r = \boldsymbol{\Omega}^L$.

5. WORK CONJUGATE AND CONJUGATE STRESSES

According to Hill (1978), the stress $\mathbf{T}$ conjugate to the generalized strain $\mathbf{E}$ can be defined through

$$w = \mathrm{III}\ \mathrm{tr}\ (\boldsymbol{\sigma}\mathbf{D}) = \mathrm{tr}\ (\mathbf{T}\dot{\mathbf{E}}), \tag{60}$$

where w represents the stress power worked on a volume element in reference configuration and $\boldsymbol{\sigma}$ the Cauchy stress. In fact, the symmetric tensor $\mathbf{T}$ in (60) can be determined uniquely as the form of $\mathbf{E}$ is prescribed.

Before deriving the general relation between $\mathbf{T}$ and $\boldsymbol{\sigma}$. Consider, as an example, the simple case $\mathbf{E} = \mathbf{E}^{(1/2)} = \mathbf{U} - \mathbf{I}$. From (60), we obtain

$$T^{(1,2)} = \frac{III}{2}(R^T \sigma R U^{-1} + U^{-1} R^T \sigma R). \tag{61}$$

This result can be found in Hill (1978). Conversely σ can be expressed in $T^{(1,2)}$. In fact, from (39) and (60) it follows that

$$\sigma = \frac{1}{III(I\,II - III)} R\{I\,III T^{(1/2)} + (I^2 + II)UT^{(1,2)}U + U^2 T^{(1,2)}U^2 - III(T^{(1,2)}U + UT^{(1/2)})$$
$$- I(UT^{(1,2)}U^2 + U^2 T^{(1/2)}U)\}R^T. \tag{62}$$

The general relations between T and σ are not as simple as (61) and (62).

Because we already have an explicit representation of $(\ln U)^*$ and E, the representation of the stress T conjugate to E can be derived without any difficulty, from (54) and (60), it follows that

$$III\,tr(\sigma \cdot D) = III\,tr(\hat{\sigma} \cdot \hat{D}) = tr[Tg(U)U(\ln U)^* + (ET - TE)\Omega^L], \tag{63}$$

where $ET - ET$ is measure-invariant (Hill, 1975, unpublished). Since $(\ln U)^*$ is measure-invariant, the diagonal components of $Tg(U)$ in Lagrangian triad and $tr(Tg(U))$, $tr(Tg(U)U)$ as well as $tr(Tg(U)U^2)$ are measure-invariant, too. Combining (42b), (54) and (63), the expression for σ by virtue of T can be obtained. For convenience we introduce the following notation,

$$T_q = Tg(U)U, \quad T_E = TE - ET + \tfrac{1}{2}(UT_q U^{-1} - U^{-1}T_q U), \tag{64}$$

then we arrive at

$$III\,R^T \sigma R = Sym\left\{T_q + \sum_{i=-1}^{1} p_i \sigma_i'\right\}, \tag{65a}$$

or

$$\sigma = \frac{1}{III} R\,Sym\left\{T_q + \sum_{i=-1}^{1} p_i \sigma_i'\right\}R^T, \tag{65b}$$

where p_i can be found in (42b) and

$$\sigma_{-1}' = T_E U^{-1} - U^{-1} T_E,$$
$$\sigma_0' = UT_E U^{-1} - U^{-1} T_E U,$$
$$\sigma_1' = UT_E - T_E U,$$

where Sym $\{\cdot\}$ means the symmetric part of $\{\cdot\}$. When the specific form of E is given in terms of U, the above representation (65a) with (65b) can be simplified further. To show this, let's consider the case $T = T^{(0)}$, from (64)–(65) we have

$$T_q = T^{(0)}, \quad T_E = \tfrac{1}{2}(UT^{(0)}U^{-1} - U^{-1}T^{(0)}U) - ((\ln U)T^{(0)} - T^{(0)}(\ln U)),$$

then

$$\sigma = \frac{1}{\text{III}} \mathbf{R} \left\{ \mathbf{T}^{(0)} + \sum_{i=-1}^{1} p_i \sigma_i^f \right\} \mathbf{R}^T. \tag{66}$$

The representation of $\mathbf{T}$ with respect to σ or $\hat{\sigma}$ can be derived from (73) through lengthy algebraic calculation.

To avoid this complexity, an alternative way can be taken for eqn (6) by applying the general approach as described in detail by Wang and Duan (1989), based on which the solution of (6) can be expressed by

$$\Omega^L = \sum_{i=-1}^{1} \omega_i^f \Omega_i^f, \tag{67}$$

where

$$\Omega_1^f = \mathbf{E}\dot{\mathbf{E}} - \dot{\mathbf{E}}\mathbf{E}, \quad \Omega_0^f = \mathbf{E}\dot{\mathbf{E}}\mathbf{E}^{-1} - \mathbf{E}^{-1}\dot{\mathbf{E}}\mathbf{E}, \quad \Omega_{-1}^f = \dot{\mathbf{E}}\mathbf{E}^{-1} - \mathbf{E}^{-1}\dot{\mathbf{E}}, \tag{68}$$

with

$$\omega_{-1}^f = \frac{\text{III}_f}{\Delta_f}(\text{I}_f^3 - 4\,\text{I}_f\,\text{II}_f + 9\,\text{III}_f), \quad \omega_0^f = \frac{\text{III}_f}{\Delta_f}(3\,\text{II}_f - \text{I}_f^2), \quad \omega_1^f = \frac{1}{\Delta_f}(\text{II}_f^2 - 3\,\text{I}_f\,\text{III}_f),$$

$$\Delta_f = 4\,\text{I}_f^3\,\text{III}_f - \text{I}_f^2\,\text{II}_f^2 - 18\,\text{I}_f\,\text{II}_f\,\text{III}_f + 4\,\text{II}_f^3 + 27\,\text{III}_f^2,$$

where I_f, II_f and III_f are the three invariants of $\mathbf{E}$. We make use of eqns (53) and (54) to express $\mathbf{D}$ in terms of $\mathbf{E}$, $\dot{\mathbf{E}}$, $\mathbf{U}$ as well as Ω^L in the form

$$\hat{\mathbf{D}} = [g(\mathbf{U})\mathbf{U}]^{-1}(\dot{\mathbf{E}} - \Omega^L\mathbf{E} + \mathbf{E}\Omega^L) - \tfrac{1}{2}(\mathbf{U}\Omega^L\mathbf{U}^{-1} - \mathbf{U}^{-1}\Omega^L\mathbf{U}). \tag{69}$$

Then eqn (60) can be rewritten as follows:

$$\text{tr}\,(\mathbf{T} \cdot \dot{\mathbf{E}}) = \text{III}\,\text{tr}\,(\sigma \cdot \mathbf{D}) = \text{III}\,\text{tr}\,(\hat{\sigma} \cdot \hat{\mathbf{D}})$$
$$= \text{III}\,\text{tr}\,\{\hat{\sigma}[(g(\mathbf{U})\mathbf{U})^{-1}(\dot{\mathbf{E}} - \Omega^L\mathbf{E} + \mathbf{E}\Omega^L) - \tfrac{1}{2}(\mathbf{U}\Omega^L\mathbf{U}^{-1} - \mathbf{U}^{-1}\Omega^L\mathbf{U})]\}, \tag{70}$$

where $\hat{\sigma} = \mathbf{R}^T\sigma\mathbf{R}$ and $\hat{\mathbf{D}} = \mathbf{R}^T\mathbf{DR}$. Now utilizing (67), inserting it into (70) and through a cumbersome algebraic calculation we finally find the solution of (70) as given by

$$\mathbf{T} = \text{III}\,\text{Sym}\left\{\sigma[g(\mathbf{U})\mathbf{U}]^{-1} + \sum_{i=-1}^{1} \omega_i^f \Sigma_i^f\right\}, \tag{71}$$

where

$$\Sigma_{-1}^f = \Sigma^f\mathbf{E}^{-1} - \mathbf{E}^{-1}\Sigma^f, \quad \Sigma_0^f = \mathbf{E}\Sigma^f\mathbf{E}^{-1} - \mathbf{E}^{-1}\Sigma^f\mathbf{E}, \quad \Sigma_1^f = \mathbf{E}\Sigma^f - \Sigma^f\mathbf{E} \tag{72}$$

with

$$\Sigma^f = -\mathbf{E}\hat{\sigma}[g(\mathbf{U})\mathbf{U}]^{-1} + \hat{\sigma}[g(\mathbf{U})\mathbf{U}]^{-1}\mathbf{E} + \tfrac{1}{2}[\mathbf{U}\hat{\sigma}\mathbf{U}^{-1} - \mathbf{U}^{-1}\hat{\sigma}\mathbf{U}]. \tag{73}$$

As an important example of the applications of (74), let us calculate the stress $\mathbf{T}^{(0)}$ conjugate to the logarithmic strain $\mathbf{E}^{(0)} = \ln \mathbf{U}$. Since $f(\lambda) = \ln \lambda$, so $\lambda f'(\lambda) = g(\lambda)\lambda = 1$, which means $g(\mathbf{U})\mathbf{U} = \mathbf{I}$. In this case, eqns (71)–(73) lead to the simpler form

$$T^{(0)} = III\left\{\dot{\sigma} + \sum_{i=-1}^{1} \omega_i^{\ln}\Sigma_i^{\ln}\right\} \tag{74}$$

with

$$\Sigma_{-1}^{\ln} = \Sigma^{\ln}(\ln U)^{-1} - (\ln U)^{-1}\Sigma^{\ln},$$

$$\Sigma_0^{\ln} = (\ln U)\Sigma^{\ln}(\ln U)^{-1} - (\ln U)^{-1}\Sigma^{\ln}(\ln U),$$

$$\Sigma_1^{\ln} = (\ln U)\Sigma^{\ln} - \Sigma^{\ln}(\ln U) \tag{75}$$

and

$$\Sigma^{\ln} = \dot{\sigma}\ln U - \ln U\dot{\sigma} + \tfrac{1}{2}(U\dot{\sigma}U^{-1} - U^{-1}\dot{\sigma}U). \tag{76}$$

This result can be compared with Hoger (1987).

With the help of (74), we can easily find the stress $\hat{T}^{(0)}$ conjugate to the strain $\ln V$. Before deriving this stress, it is worth pointing out that since $(\ln V)^{\cdot}$ is not objective, an objective time derivative, say $(\ln V)^{\circ}$, has to replace $(\ln V)^{\cdot}$ in using the formula (60), that is, we must have

$$III\, \mathrm{tr}\,(\sigma D) = \mathrm{tr}\,(T^{(0)}(\ln U)^{\cdot}); \quad \mathrm{tr}\,(\sigma D) = \mathrm{tr}\,(\hat{T}^{(0)}(\ln V)^{\circ}). \tag{77}$$

Since $(\ln U)^{\cdot} = R^{T}(\ln V)^{\circ}R$, from (85) we immediately arrive at

$$\hat{T}^{(0)} = \frac{1}{III}RT^{(0)}R^{T}, \quad T^{(0)} \to \hat{T}^{(0)} = III\,\hat{T}^{(0)}. \tag{78}$$

Therefore, the conjugate stress $\hat{T}^{(0)}$ of $\ln V$ is just $\hat{T}^{(0)}/III$, the mapping of the conjugate stress $T^{(0)}$ of $\ln U$ in the current configuration by using the transformation R as defined in (25), divided by III. The same conclusion can be drawn for the relation between the stress conjugate to $f(U)$ and that for $f(V)$.

Hoger (1987) discussed the conjugate stress of $\ln V$ and concluded that the stress generally does not exist. However she seems to ignore the fact that the generalized strain E and its conjugate stress T are defined in the reference configuration. If use is made of the work conjugate principle to define the conjugate stresses in the current configuration, an objective time derivative of strain should be defined to replace the simple time derivative of the strain. The above discussion concerning the conjugate stress for $\ln V$ obviously confirms our argument.

6. CONCLUDING REMARKS

Making use of a group of complete and irreducible generators of a linear tensorial equation, the solution of this equation is obtained by an expansion technique. In particular, the tensorial equation for a spin Ω^L of the principal triad of a symmetric and non-degenerate tensor U has been formulated and discussed in detail. Based on this novel technique, the invariant representation of the spin in terms of U and $\dot{U}$ has been given in a closed form, and it has a simpler form as the tensor U has multiple eigenvalues.

To show the applications of the proposed method, three specific problems which are obviously very fundamental in the study of constitutive relations for finite deformations have been successfully treated, including the stretch tensor and its spin, the invariant representations of generalized strain rate. Above all, the general invariant representation of the stress and the generalized strain, which, to the best of the authors' knowledge, has not been known before. As an important example of the applications, the logarithmic strain and its conjugate stress given either in the reference configuration or in the current

configuration have been discussed in some detail and their invariant representations have been analytically formulated in a compact form.

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